JOURNAL OF APPROXIMATION THEORY 60, 70–82 (1990)

On the Fundamental Lemma of Interpolation Theory

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Communicated by P. L. Butzer

Received January 11, 1988

1. INTRODUCTION

A basic issue in real interpolation theory is the relationship between the K and J methods of interpolation as applied to compatible couples of Banach spaces or Banach couples $\bar{A}=(A_0, A_1)$. The important classical result that the spaces $\bar{A}_{\theta,q}$ can be obtained equivalently by either of these methods is proved with the help of the so-called "Fundamental Lemma of Interpolation Theory" (cf. [BL]).

(1.0) LEMMA. *If* $\bar{A}=(A_0, A_1)$ *is a Banach couple and* $a \in \Sigma(\bar{A})$ *is such that* $\min\{1, 1/t\}\, K(t, a; \bar{A})$ *tends to zero as* t *tends to* 0 *or* ∞, *then for each* $\varepsilon > 0$ *there is a representation of* a *such that*

$$(1.1) \qquad a=\sum_{n=-\infty}^{\infty} u_n, \qquad u_n \in \Delta(\bar{A}) \text{ (convergence in } \Sigma(\bar{A}))$$

* Research supported in part by the Fund for Promotion of Research at Technion.
† Supported in part by an NSF grant.

0021-9045/90 $3.00

and

$$J(2^n, u_n, \bar{A}) \leqslant (\alpha+\varepsilon)\, K(2^n, a; \bar{A}) \qquad \text{for all} \quad n \in \mathbb{Z}. \tag{1.2}$$

Here α is an absolute constant satisfying $\alpha \leqslant 3$.

In their remarkable paper [BK1] Ju. A. Brudnyĭ and N. Ja. Krugljak obtained the following result ("K-divisibility of Peetre's K-functional"):

(1.3) THEOREM. *Let $\bar{A}$ be a Banach couple and let a be any element of $\Sigma(\bar{A})$. Suppose that $K(t, a, \bar{A}) \leqslant \sum_{n=1}^{\infty} \phi_n(t)$ for all $t>0$, where each ϕ_n is a positive concave function on $(0, \infty)$ and $\sum_{n=1}^{\infty} \phi_n(t) < \infty$. Then there exists a sequence $\{a_n\}$ of elements in $\Sigma(\bar{A})$ such that $a = \sum_{n=1}^{\infty} a_n$ (convergence in $\Sigma(\bar{A})$) and $K(t, a_n; \bar{A}) \leqslant \gamma\phi_n(t)$ for some constant $\gamma \leqslant 14$ and all $t>0$.*

If we use the fact (cf., e.g., [BL], Lemma 5.4.3, p. 117] or [C, Lemma 1, p. 46]) that $K(t, a; \bar{A})$ is equivalent to a function of the form $\sum_{n=-\infty}^{\infty} \lambda_n \min(1, t/(1+\varepsilon)^n)$ when the element a satisfies the hypotheses of Lemma (1.0), then it can readily be shown that Theorem (1.3) leads to the following stronger version of the fundamental lemma for all mutually closed Banach couples (see Section 2 for precise definitions):

(1.4) THEOREM. *Let $\bar{A}$ be a mutually closed couple of Banach spaces and let $a \in \Sigma(\bar{A})$ be such that $\min\{1, 1/t\}\, K(t, a; \bar{A}) \to 0$ as $t \to 0$ or $t \to \infty$ (i.e., a belongs to $\Sigma^{\circ}(\bar{A})$, the closure of $\Delta(\bar{A})$ in $\Sigma(\bar{A})$). Then there exist elements $u_n \in \Delta(\bar{A})$ for each $n \in \mathbb{Z}$ such that*

$$a = \sum_{n=-\infty}^{\infty} u_n \qquad (\textit{convergence in } \Sigma(\bar{A}) \textit{ norm}) \tag{1.5}$$

and, moreover, there exists a constant $\gamma \leqslant 14$ such that

$$\sum_{n=-\infty}^{\infty} \min\{1, t/2^n\}\, J(2^n, u_n; \bar{A}) \leqslant \gamma K(t, a; \bar{A}). \tag{1.6}$$

We shall refer to (1.4) and its variants as forms of "The Strong Fundamental Lemma" or SFL for short. We also note that if $\bar{A}$ is mutually closed and $a \in \Sigma^{\circ}(\bar{A})$ then (1.4) in fact implies (1.3).

In [C] one of us obtained and used the following variant of the SFL which is strong enough to imply the K-divisibility property and showed moreover that the constant γ can be taken to be any number greater than 8, though not necessarily 8 itself.

(1.7) THEOREM. *Let $\bar{A}$ be a Banach couple and let $\bar{B}$ denote the couple (B_0, B_1), where B_j is the Gagliardo closure of A_j in $\Sigma(\bar{A})$, $j = 0, 1$. Then for* *each* $\varepsilon > 0$ *there exists a sequence of elements* $\{u_{n,\varepsilon}\} = \{u_n\}$ *in* $\Sigma(\bar{A})$ *such that:*

(i) $u_n \in \Delta(\bar{A})$ *for all but at most two values of* n *and, moreover,* $\sum_{n=-\infty}^{\infty} u_n = a$ *(convergence in* $\Sigma(\bar{A})$ *norm),*

(ii) $\sum_{n=-\infty}^{\infty} \min\{\|u_n\|_{B_0}, t\,\|u_n\|_{B_1}\} \leqslant \beta(1+\varepsilon)\,K(t, a; \bar{A})$ *for all* $t > 0$, *where* $\beta \leqslant 8$. *(In the preceding estimate we set* $\|u_n\|_{B_j} = \infty$ *if* $u_n \notin B_j$.)

Our goal in this paper is to reexamine the SFL in order to refine its proof and present some other versions of it. Our motivation for doing this comes from the fact that the SFL, together with the essentially equivalent K-divisibility theorem, has become an important tool in recent developments in interpolation theory. For example, Brudnyĭ and Krugljak use it in [BK1] to study the so-called *K spaces.* One reason why these spaces are of interest is that for many Banach couples, so-called *Calderón–Mityagin* couples, all the interpolation spaces with respect to the couple are K spaces. Brudnyĭ and Krugljak are able to show that although K spaces are defined in terms of a rather abstract monotonicity condition, they in fact always have a much more concrete structure, their norms being given by lattice norms acting on the K-functionals of the elements of the spaces (cf. also [CP]). They also give very general conditions for equivalence to hold between K and J spaces. In [N1] Nilsson provides, among many other things, interesting applications of the SFL to the computation of K-functions for "classical" Banach couples. Using K-divisibility or the SFL it is possible to establish some rather non-trivial properties of the K-functional for a couple of Banach lattices. (See [CN] or [CS] and [BK2].) Furthermore a special form of the SFL for couples of lattices gives new insights into mechanisms which cause certain couples to be Calderón–Mityagin couples, enabling previous results about these couples to be considerably unified and extended [C, CN]. One would be inclined to assume that the only interpolation spaces for which SFL can be relevant are those obtained by the K and J methods. However, surprisingly enough, it can also be used to yield new results concerning very different spaces, such as those of Calderón–Lozanovskii, Gustavsson–Peetre, or Ovcinnikov. This was shown by Nilsson in [N2]. Finally, in this list of applications we refer to an example mentioned in [BK1] indicating the possibility of decomposing a function in a very delicate way with respect to its modulus of continuity. Indeed this last example hints at possible applications rather beyond the usual framework of interpolation theory.

We now describe the contents of the present paper in more detail: In Section 2 we provide a new proof of (1.7) which gives the best known value of the constant β for which (1.7)(ii) holds, namely $\beta = 3 + 2\sqrt{2}$ as announced by A. A. Dmitriev [D]. Our proof is a refinement of the one given in [C] and is based on a careful analysis of the relationship between Peetre's K-functionals and Gagliardo's diagrams. To date no details concerning Dmitriev's proof have reached us, despite the time which has elapsed since its announcement. We have at least one trivial reason to suspect that our proof may be different from Dmitriev's, namely that the constant in [D] is written in the form $(1+\sqrt{2})^2$. Of course the problem of finding the best constants for all forms of the fundamental lemmas of interpolation theory remains open. We remark that this problem is not new. In particular Peetre [P] formulated and discussed it several years ago for the "classical" case.

Our result implies, via the analysis of [C], that the K-divisibility constant in Theorem (1.3) can also be taken to be less than or equal to $(3+2\sqrt{2})(1+\varepsilon)$.

In Section 3 we consider a variant of (1.7), namely a "continuous" version which also implies (1.6). In Section 4 we formulate a variant of the SFL for the E method of interpolation. Results of this type have also been considered, though in less general form, in [N1]. One could expect them to be useful in the theory of ideals of operators (cf. [K]) although we do no explore this issue here.

The notation and terminology which we use are essentially the same as in [BL].

2. A Proof of the SFL

In this section we provide a proof of (1.7) with constant $\beta = 3+2\sqrt{2}$.

We start by briefly discussing the relationship between Gagliardo diagrams and the K-functional. The computation of the K-functional can be given a geometrical interpretation as follows: For each $a \in \Sigma(\bar{A})$ we define the *Gagliardo diagram* of a to be the set

$$\Gamma(a) = \{(x_0, x_1) \colon \exists a_j \in A_j \text{ s.t. } \|a_j\|_{A_j} \leqslant x_j,\ j=0,1,\ a=a_0+a_1\}.$$

It is easy to see that $\Gamma(a)$ is a convex set. Its boundary may contain a semi-infinite vertical segment and/or a semi-infinite horizontal segment. The remainder of the boundary will be the graph of a decreasing convex function $x_1 = \phi(x_0)$. Let

$$D(a) = \partial\Gamma(a) \cap \{(x_0, x_1) \in \mathbb{R}^2 \colon x_j > 0,\ j=0,1\}.$$

It follows (cf. [BL]) that for each $t>0$, $K(t, a; \bar{A}) = K(t)$ is the x_0 intercept of the tangent to $D(a)$ with slope $-1/t$. Thus the corresponding x_1 intercept is of course $K(t)/t$. Conversely, each point (x_0, x_1) on the graph of ϕ intersects with the tangent of slope $-1/t$ for some value(s) of t determined by the left and right derivatives of ϕ at x_0 and thus, of course, $x_0 + tx_1 = K(t, a; \bar{A})$.

The *Gagliardo completion* of A_j, $j = 0, 1$, which we denote by $\tilde{A}_j$, is the set of elements a of $\Sigma(\bar{A})$ which are $\Sigma(\bar{A})$ limits of bounded sequences in A_j or, equivalently, for which $\|a\|_{\tilde{A}_j} = \sup_{t>0} K(t)/t^j$ is finite. We say that the couple $\bar{A}$ is *mutually closed* if $A_j = \tilde{A}_j$ for $j = 0, 1$ with equality of norms. (In previous papers using this notion it has not always been clearly specified whether the norms must be equal or merely equivalent. We refer to [CP] and also [CN] for examples and more details.)

We define

$$\begin{aligned} x_\infty &= \sup\{x\colon (x, y) \in D(a)\}, & x_{-\infty} &= \inf\{x\colon (x, y) \in D(a)\} \\ y_{-\infty} &= \inf\{y\colon (x, y) \in D(a)\}, & y_\infty &= \sup\{y\colon (x, y) \in D(a)\}. \end{aligned} \tag{2.1}$$

From the geometrical considerations above it follows that

$$\begin{aligned} x_\infty &= \lim_{t \to \infty} K(t) = \|a\|_{\tilde{A}_0}, & x_{-\infty} &= \lim_{t \to 0} K(t) \\ y_{-\infty} &= \lim_{t \to \infty} K(t)/t, & y_\infty &= \lim_{t \to 0} K(t)/t = \|a\|_{\tilde{A}_1}. \end{aligned} \tag{2.2}$$

Now we construct a sequence of points $\{(x_n, y_n)\}_{\nu_{-\infty} - 1 < n < \nu_\infty + 1}$ lying on $D(a)$. (Cf. [G1, p. 227; G2, p. 95].) Fix any point (x_0, y_0) on $D(a)$ and let $r = 1 + \sqrt{2}$. For each $n > 0$ construct (x_n, y_n) inductively such that

$$\text{either} \begin{cases} x_n = r x_{n-1} \\ y_n \leqslant \dfrac{1}{r}\, y_{n-1} \end{cases} \text{or} \begin{cases} x_n \geqslant r x_{n-1} \\ y_n = \dfrac{1}{r}\, y_{n-1} \end{cases} \tag{2.3}$$

holds. This construction must stop if for some n we have

$$\text{either } r x_{n-1} \geqslant x_\infty \text{ or } \frac{1}{r}\, y_{n-1} \leqslant y_{-\infty} \tag{2.4}$$

in which case we define $\nu_\infty = n$, and do not need to define x_n or y_n. Alternatively if (2.4) does not hold for any positive n then we set $\nu_\infty = \infty$.

In a similar way for $n < 0$ we go "backwards" and inductively construct a sequence (x_n, y_n) such that

$$\text{either} \begin{cases} x_n = \dfrac{1}{r}\, x_{n+1} \\ \dfrac{1}{r}\, y_n \leqslant y_{n+1} \end{cases} \text{or} \begin{cases} x_n \leqslant \dfrac{1}{r}\, x_{n+1} \\ \dfrac{1}{r}\, y_n = y_{n+1} \end{cases} \tag{2.5}$$

holds. The construction must stop if for some $n<0$

$$\text{either } \frac{1}{r}x_{n+1} \leqslant x_{-\infty} \text{ or } ry_{n+1} \geqslant y_{\infty} \tag{2.6}$$

holds, in which case we set $\nu_{-\infty}=n$ and do not need to define x_n and y_n. Otherwise we set $\nu_{-\infty}=-\infty$.

Now, given $\varepsilon>0$, we can find a decomposition $a=a_n+a_n'$ for each n, $\nu_{-\infty}<n<\nu_{\infty}$, such that

$$\begin{aligned} x_n &\leqslant \|a_n\|_{A_0} \leqslant (1+\varepsilon)x_n \\ y_n &\leqslant \|a_n'\|_{A_1} \leqslant (1+\varepsilon)\,y_n. \end{aligned} \tag{2.7}$$

We define the sequence $\{u_n\}$ as

$$u_n=\begin{cases} a_n-a_{n-1}=a_{n-1}'-a_n' & \text{if } \nu_{-\infty}+1<n<\nu_{\infty} \\ a-a_{\nu_\infty-1}=a_{\nu_\infty-1}' & \text{if } n=\nu_\infty<\infty \\ a-a_{\nu_{-\infty}+1}'=a_{\nu_{-\infty}+1} & \text{if } n=\nu_{-\infty}+1>-\infty \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Observe that $\sum_{n=-\infty}^{\infty}u_n=a$, where the series converges in $\Sigma(\bar{A})$ norm. In fact, if $\nu_{-\infty}>-\infty$, then $\sum_{n=-\nu_{-\infty}}^{0}u_n=a_0$, and if $\nu_{-\infty}=-\infty$ then $\|\sum_{n=-\nu_{-\infty}}^{0}u_n-a_0\|_{A_0} = \lim_{n\to\infty}\|a_n\|_{A_0} \leqslant (1+\varepsilon)\lim_{n\to\infty}x_n = (1+\varepsilon)\lim_{n\to\infty}(x_0/r^n)=0$. Similarly, $\sum_{n=1}^{\infty}u_n=a_0'$ with convergence in A_1 norm, whether or not ν_∞ is finite.

As a first step to proving (1.7)(ii) we need some preliminary estimates for $\|u_n\|_{A_0}$ and $\|u_n\|_{A_1}$. If $\nu_{-\infty}+1<n<\nu_\infty$ then

$$\|u_n\|_{A_0}=\|a_n-a_{n-1}\|_{A_0}\leqslant\|a_n\|_{A_0}+\|a_{n-1}\|_{A_0}\leqslant(1+\varepsilon)(x_n+x_{n-1})$$

and so

$$\|u_n\|_{A_0}\leqslant(1+\varepsilon)(1+1/r)\,x_n. \tag{2.9}$$

Observe that (2.9) trivially holds also for $n=\nu_{-\infty}+1$. Similarly $\|u_n\|_{A_1}\leqslant(1+\varepsilon)(y_n+y_{n-1})$ for $\nu_{-\infty}+1<n<\nu_\infty$ and so

$$\|u_n\|_{A_1}\leqslant(1+\varepsilon)(1+1/r)\,y_{n-1} \tag{2.10}$$

for these values of n and in fact also for $n=\nu_\infty$ since, by (2.8) and (2.7), $\|u_{\nu_\infty}\|_{A_1}=\|a_{\nu_\infty-1}'\|_{A_1}\leqslant(1+\varepsilon)\,y_{\nu_\infty-1}$.

Now for the proof of (1.7)(ii) for any given value of $t>0$ we must consider three different cases. It will be convenient to let Λ_t denote the set of all points on the tangent to $\Gamma(a)$ having slope $-1/t$. Suppose first that there exists an integer n^*, $\nu_{-\infty}+1<n^*<\nu_\infty$, and also a point $(x^*, y^*)\in D(a)\cap A_t$ such that (x^*, y^*) lies "between" (x_{n^*-1}, y_{n^*-1}) and (x_{n^*}, y_{n^*}), i.e., $x_{n^*-1}\leqslant x^*\leqslant x_{n^*}$ and $y_{n^*-1}\geqslant y^*\geqslant y_{n^*}$. Then in order to estimate the sum $\sum_{n=-\infty}^{\infty}\min\{\|u_n\|_{A_0}, t\,\|u_n\|_{A_1}\}=\sum_{n=-\infty}^{\infty} m_n$, we first note that by (2.9)

$$\begin{aligned} I_- = \sum_{n=-\infty}^{n^*-1} m_n &\leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right)\sum_{n=\nu_{-\infty}+1}^{n^*-1} x_n \\ &\leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right)\sum_{n=\nu_{-\infty}+1}^{n^*-1}\left(\frac{1}{r}\right)^{n^*-1-n} x_{n^*-1} \\ &\leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right)\left(1-\frac{1}{r}\right)^{-1} x_{n^*-1}. \end{aligned}$$

Then similarly, by (2.10),

$$I_+ = \sum_{n=n^*+1}^{\nu_\infty} m_n \leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right)\left(1-\frac{1}{r}\right)^{-1} t y_{n^*}.$$

Moreover, since $x_{n^*-1}+ty_{n^*}\leqslant x^*+ty^*=K(t)$, we can combine the two preceding estimates to obtain

$$I_- + I_+ \leqslant (1+\varepsilon)(r+1)(r-1)^{-1} K(t). \tag{2.11}$$

We can also see that $m_{n^*}\leqslant(1+\varepsilon)(1+r)\,K(t)$ since either $rx_{n^*-1}=x_{n^*}$ holds, in which case

$$m_{n^*}\leqslant \|u_{n^*}\|_{A_0}\leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right) r x_{n^*-1}\leqslant (1+\varepsilon)(1+r)\,K(t),$$

or otherwise $ry_{n^*}=y_{n^*-1}$ must hold and then

$$m_{n^*}\leqslant t\,\|u_{n^*}\|_{A_1}\leqslant (1+\varepsilon)\left(1+\frac{1}{r}\right) t r y_{n^*}\leqslant (1+\varepsilon)(1+r)\,K(t).$$

Combining the estimate for m_{n^*} with (2.11) we obtain that

$$\sum_{n=-\infty}^{\infty}\min\{\|u_n\|_{A_0}, t\,\|u_n\|_{A_1}\}\leqslant (1+\varepsilon)(3+2\sqrt{2})\,K(t)$$

as required.

It remains to consider the possibility that there does not exist an integer n^* with the above properties for the value of t under consideration. This means that either $\nu_\infty<\infty$ and $x^*\geqslant x_{\nu_\infty-1}$ for all points

$(x^*, y^*) \in \Lambda_t \cap D(a)$, or, alternatively, $\nu_{-\infty} > -\infty$ and $x^* \leqslant x_{\nu_{-\infty}+1}$ and $y^* \geqslant y_{\nu_{-\infty}+1}$ for all points $(x^*, y^*) \in \Lambda_t \cap D(a)$. In the first of these cases, in order to estimate $\sum_{n=-\infty}^{\nu_\infty} \min\{\|u_n\|_{\tilde{A}_0}, t\,\|u_n\|_{\tilde{A}_1}\} = \sum_{n=-\infty}^{\nu_\infty} \tilde{m}_n$, we first observe that, exactly as before,

$$I_- = \sum_{n=-\infty}^{\nu_\infty - 1} \tilde{m}_n \leqslant (1+\varepsilon)(r+1)(r-1)^{-1} x_{\nu_\infty - 1}$$

and therefore $I_- \leqslant (1+\varepsilon)(r+1)(r-1)^{-1} K(t)$. Then to estimate the only remaining non-zero term $\tilde{m}_{\nu_\infty}$ we must again consider two possibilities: Either $r x_{\nu_\infty - 1} \geqslant x_\infty$ holds, in which case

$$\begin{aligned}\tilde{m}_{\nu_\infty} &\leqslant \|u_{\nu_\infty}\|_{\tilde{A}_0} \leqslant \|a\|_{\tilde{A}_0} + \|a_{\nu_\infty - 1}\|_{\tilde{A}_0} \leqslant x_\infty + (1+\varepsilon) x_{\nu_\infty - 1} \\ &\leqslant (r+1)(1+\varepsilon) x_{\nu_\infty - 1} \leqslant (1+\varepsilon)(r+1)\, K(t),\end{aligned}$$

or the remaining possibility is that $(1/r)\, y_{\nu_\infty - 1} \leqslant y_{-\infty}$ holds and in that case

$$\tilde{m}_{\nu_\infty} \leqslant t\,\|a'_{\nu_\infty - 1}\|_{\tilde{A}_1} \leqslant t(1+\varepsilon)\, y_{\nu_\infty - 1} \leqslant t(1+\varepsilon) r y_{-\infty} \leqslant (1+\varepsilon)\,\mathrm{tr}\, K(t)/t.$$

Thus in both cases we obtain that

$$\sum_{n=-\infty}^{\nu_\infty} \tilde{m}_n \leqslant (1+\varepsilon)(r+1)(1+(r-1)^{-1})\, K(t) \leqslant (1+\varepsilon)(3+2\sqrt{2})\, K(t).$$

An analogous argument, whose details we leave to the reader, gives a similar estimate in the case when $\nu_{-\infty} > -\infty$. Thus we have shown that (1.7)(ii) holds in all cases with $\beta = 3 + 2\sqrt{2}$ and so completed the proof of the SFL.

(2.12) *Remarks.* Observe that by construction $u_n \in \Delta(\bar{A})$, except possibly when $n = \nu_{-\infty} + 1$ or $n = \nu_\infty$. Let us also point out that if $x_{-\infty} = 0$ and $y_{-\infty} = 0$, then $u_n \in \Delta(\bar{A})$ for all n. For example, if $\nu_\infty < \infty$ and $y_{-\infty} = 0$, then x_∞ is finite and $\|u_{\nu_\infty}\|_{\tilde{A}_0} \leqslant (2+\varepsilon) x_\infty$.

It will be convenient in the sequel to refer to the following variant of the SFL which is an immediate corollary of the above remarks and the preceding proof of Theorem (1.7).

(2.13) THEOREM. *Let $\bar{A}$ be a mutually closed couple of Banach spaces and let $a \in \Sigma^\circ(\bar{A})$. Then for each positive ε there exist elements $u_{n,\varepsilon} = u_n \in \Delta(\bar{A})$ for each $n \in \mathbb{Z}$ such that:*

(i) $a = \sum_{n=-\infty}^{\infty} u_n$ *(convergence in $\Sigma(\bar{A})$ norm) and*

(ii) $\sum_{n=-\infty}^{\infty} \min\{\|u_n\|_{A_0}, t\,\|u_n\|_{A_1}\} \leqslant (1+\varepsilon)(3+2\sqrt{2})\, K(t).$

3. A Continuous Variant of the SFL

We observe from the result in this section that it is possible to construct "continuous" representations which simultaneously have the property guaranteed by the continuous form of the "classical" fundamental lemma as well as two forms of the analogous "strong" property.

(3.1) Theorem. *Let $\bar{A}$ be a mutually closed Banach couple, and let $a \in \Sigma^{\circ}(\bar{A})$. Then for each positive ε there exists a $\Delta(\bar{A})$ valued function $u = u_\varepsilon$ on $(0, \infty)$ which is strongly measurable on each subinterval (ξ, η), $\xi, \eta > 0$, such that $a = \int_0^\infty u(t)(dt/t)$ (considered as the integral of a $\Sigma(\bar{A})$ valued function) and such that each of the following three conditions is satisfied for all $t > 0$:*

$$\int_0^\infty \min\{1, t/s\}\, J(s, u(s); \bar{A}) \frac{ds}{s} \leqslant (1+\varepsilon)^2 \beta K(t) \tag{3.2}$$

$$\int_0^t \|u(s)\|_{A_0} \frac{ds}{s} + t \int_t^\infty \|u(s)\|_{A_1} \frac{ds}{s} \leqslant (1+\varepsilon)^2 \beta K(t) \tag{3.3}$$

$$\int_0^\infty \min\{\|u(s)\|_{A_0}, t\, \|u(s)\|_{A_1}\} \frac{ds}{s} \leqslant (1+\varepsilon)^2 \beta K(t) \tag{3.4}$$

$$J(t, u(t); \bar{A}) \leqslant (1+\varepsilon)^3 [\log(1+\varepsilon)]^{-1} \beta K(t) \qquad \text{for all} \quad t > 0, \tag{3.5}$$

where, as usual, $K(t) = K(t, a; \bar{A})$ and β is the constant appearing in the SFL.

Proof. We begin by applying Theorem (2.13) to obtain a representation $\sum_{n=-\infty}^{\infty} u_n$ of a satisfying (2.13)(i)–(ii). Then for each $n \in \mathbb{Z}$ we let

$$S_n = \{v \in \mathbb{Z} : (1+\varepsilon)^n < \|u_v\|_{A_0} / \|u_v\|_{A_1} \leqslant (1+\varepsilon)^{n+1}\}$$

and $v_n = \sum_{v \in S_n} u_v$. By (2.13)(ii) this series converges absolutely in $\Delta(\bar{A})$. More specifically,

$$\|v_n\|_{A_0} \leqslant \sum_{v \in S_n} \min\{\|u_v\|_{A_0}, (1+\varepsilon)^{n+1} \|u_v\|_{A_1}\} \tag{3.6}$$

and

$$(1+\varepsilon)^n \|v_n\|_{A_1} \leqslant \sum_{v \in S_n} \min\{\|u_v\|_{A_0}, (1+\varepsilon)^n \|u_v\|_{A_1}\}. \tag{3.7}$$

Define the vector valued function $u(t)$ by

$$u(t) = \sum_{n=-\infty}^{\infty} \frac{1}{\log(1+\varepsilon)} v_n \chi_{((1+\varepsilon)^n, (1+\varepsilon)^{n+1}]}(t).$$

Obviously $u(t)$ is a strongly measurable $\Delta(\bar{A})$ valued function in each interval (ξ, η) and, as a $\Sigma(\bar{A})$ valued function, is absolutely integrable on $(0, \infty)$ and satisfies

$$\int_0^\infty u(t)\frac{dt}{t} = \sum_{n=-\infty}^{\infty} v_n = \sum_{\nu=-\infty}^{\infty} u_\nu = a.$$

Now, in order to verify (3.5), given $t > 0$ choose $n^* \in \mathbb{Z}$ such that $(1+\varepsilon)^{n^*} < t \leqslant (1+\varepsilon)^{n^*+1}$. Then, using (3.6), (3.7), and (2.13)(ii), we see that

$$\begin{aligned} J(t, u(t); \bar{A}) &\leqslant J((1+\varepsilon)^{n^*}, v_{n^*}; \bar{A})/\log(1+\varepsilon) \\ &\leqslant (1+\varepsilon)^2 \beta K((1+\varepsilon)^{n^*+1})/\log(1+\varepsilon) \\ &\leqslant (1+\varepsilon)^3 \beta K(t)/\log(1+\varepsilon) \end{aligned}$$

as required. Next, to establish (3.2) we obtain similarly that, for each integer n,

$$\begin{aligned} \int_{(1+\varepsilon)^n}^{(1+\varepsilon)^{n+1}} \min\{1, t/s\}\, J(s, u(s); \bar{A})\frac{ds}{s} &\leqslant \min(1, t/(1+\varepsilon)^n)\, J((1+\varepsilon)^{n+1}, v_n; \bar{A}) \\ &\leqslant \min(1, t/(1+\varepsilon)^n) \sum_{\nu\in S_n} \min\{\|u_\nu\|_{A_0}, (1+\varepsilon)^{n+1}\, \|u_\nu\|_{A_1}\} \\ &\leqslant (1+\varepsilon) \sum_{\nu\in S_n} \min\{\|u_\nu\|_{A_0}, t\, \|u_\nu\|_{A_1}\} \end{aligned}$$

and summation over n yields (3.2). Finally, (3.2) immediately implies (3.3), which in turn immediately implies (3.4). ∎

Remark. In fact we can also go in the opposite direction of the preceding proof and deduce the discrete version of the SFL (Theorem (2.13)) from the continuous version (Theorem (3.1)), again without any worsening of the constant β. This can be done as follows: Given a continuous representation satisfying the conditions of (3.1), a discrete one can be obtained by letting

$$B_n = \{s \in (0, \infty): (1+\varepsilon)^n < \|u(s)\|_{A_0}/\|u(s)\|_{A_1} \leqslant (1+\varepsilon)^{n+1}\}$$

for each $n \in \mathbb{Z}$, and defining $u_n = \int_{B_n} u(s)(ds/s)$. Then it is readily verified that $a = \sum_{n=-\infty}^{\infty} u_n$ is a representation satisfying (2.13)(i) and (ii). For this it is in fact sufficient to assume that $u(s)$ satisfies only (3.4) rather than the apparently stronger conditions (3.2) or (3.3).

4. An E-Functional Version of the SFL

We start by recalling some well-known facts about E-functionals and a variant of the "classical" fundamental lemma involving this functional.

For each $t>0$ the E-functional of an element $a \in \Sigma(\bar{A})$ with respect to a Banach couple $\bar{A}=(A_0, A_1)$ is defined by

$$E(t, a; \bar{A}) = \inf_{\|a_0\|_{A_0} \leqslant t} \|a-a_0\|_{A_1}. \tag{4.1}$$

Notice that $E(t, a; \bar{A})$ can be infinite, which leads us to consider the set $\Sigma_E(\bar{A})$ of all elements $a \in \Sigma(\bar{A})$ such that $E(t, a; \bar{A}) < \infty$ for all positive t. This condition is equivalent to requiring that $x_{-\infty} = \lim_{t \to 0} K(t, a; \bar{A}) = 0$ (cf. Section 2). Similarly, the condition $\lim_{t \to \infty} E(t, a; \bar{A}) = 0$ is equivalent to requiring that $y_{-\infty} = \lim_{t \to \infty} K(t, a; \bar{A})/t = 0$. We also recall that the graph of the function $t \to E(t, a; \bar{A})$ is the curve defining the (non-vertical part of the) frontier Gagliardo diagram of the element a. We say that a sequence $\{a_n\}_{n>0}$ converges to a in $\Sigma_E(\bar{A})$ as n tends to infinity if $\lim_{n \to \infty} E(t, a-a_n, \bar{A}) = 0$ for each $t>0$.

A version of the "classical" fundamental lemma in terms of the E-functional states (cf. [N1]) that if $a \in \Sigma_E(\bar{A})$ and $\lim_{t \to \infty} E(t, a; \bar{A}) = 0$, then there exists a sequence $\{u_n\}_{n \in \mathbb{Z}}$ of elements in $\Delta(\bar{A})$ such that

(i) $a = \sum_{n=-\infty}^{\infty} u_n$ (convergence in $\Sigma_E(\bar{A})$),

(ii) $\|u_n\|_{A_1} \leqslant 2^n$ for all $n \in \mathbb{Z}$,

(iii) $\|u_n\|_{A_1} \leqslant cE(2^{n-2}, a; \bar{A})$ for all $n \in \mathbb{Z}$, where $c \leqslant 4$ is an absolute constant.

We now state our version of the SFL in this context:

(4.2) Theorem. *Let $\bar{A}$ be a mutually closed Banach couple and let $a \in \Sigma_E(\bar{A})$ be such that $\lim_{t \to \infty} E(t, a; \bar{A}) = 0$. Then for each positive ε there exists a $\Delta(\bar{A})$ valued function $v = v_\varepsilon$ on $(0, \infty)$ which is strongly measurable on each subinterval (ξ, η), $\xi, \eta > 0$, such that:*

(i) *$a = \int_0^\infty v(t)(dt/t)$ (convergence in $\Sigma(\bar{A})$ and also in $\Sigma_E(\bar{A})$),*

(ii) *$\int_0^t \|v(s)\|_{A_0} (ds/s) \leqslant \beta(1+\varepsilon)t$ for all $t>0$,*

(iii) *$\int_t^\infty \|v(s)\|_{A_1} (ds/s) \leqslant \beta(1+\varepsilon) E(t, a; \bar{A})$ for all $t>0$, where β is the same constant as appears in* (2.13).

Proof. By the preceding remarks $a \in \Sigma^\circ(\bar{A})$ and so, using Theorem (3.1), we can find a representation $a = \int_0^\infty u(s)(ds/s)$ satisfying (3.3). For each $t>0$ define $b_0(t) = \int_0^t \|u(s)\|_{A_0} (ds/s)$ and $b_1(t) = \int_t^\infty \|u(s)\|_{A_1} (ds/s)$.

From (3.3) we see that for each such t the point $(b_0(t)/\beta(1+\varepsilon)^2, b_1(t)/\beta(1+\varepsilon)^2)$ lies below $\Gamma(a)$. Therefore, by definition,

$$E(b_0(t)/\beta(1+\varepsilon)^2, a; \bar{A}) \geqslant b_1(t)/\beta(1+\varepsilon)^2. \tag{4.3}$$

We let the number C be 1 if $b_0(t)=\int_0^\infty \|u(s)\|_{A_0}(ds/s)$ is infinite. Otherwise we set $C=b_0(\infty)$. Since $b_0(t)$ is continuous and $\lim_{t\to 0} b_0(t)=0$, for each negative integer n we can find $t_n>0$ such that $b_0(t_n)/(1+\varepsilon)^2\beta=(1+\varepsilon)^n C$. If $b_0(\infty)=\infty$ we can also find such a t_n for all non-negative n. We can now rewrite (4.3) as

$$E((1+\varepsilon)^n C, a; \bar{A}) \geqslant b_1(t_n)/\beta(1+\varepsilon)^2. \tag{4.4}$$

We define the elements $u\in\Delta(\bar{A})$ by $u_n=\int_{t_{n-1}}^{t_n} u(s)(ds/s)$ for all n if $b_0(\infty)=\infty$. If $b_0(\infty)<\infty$ then this definition only applies for negative n and we shall define u_0 analogously by setting $t_0=\infty$ and also take $u_n=0$ and $t_n=\infty$ for all positive n. In both cases we see that $u_n\in\Delta(\bar{A})$ for all n and $a=\sum_{n=-\infty}^{\infty} u_n$, where this sum converges in $\Sigma(\bar{A})$ norm as well as in $\Sigma_E(\bar{A})$. Moreover it follows from (4.4) that

$$\beta(1+\varepsilon)^2 E((1+\varepsilon)^n C, a; \bar{A}) \geqslant b_1(t_n) = \sum_{\nu>n} \int_{t_{\nu-1}}^{t_\nu} \|u(s)\|_{A_1} \frac{ds}{s}$$
$$\geqslant \sum_{\nu>n} \|u_\nu\|_{A_1}$$

for all $n\in\mathbb{Z}$. We also obtain that

$$\sum_{\nu\leqslant n} \|u_\nu\|_{A_0} \leqslant \sum_{\nu\leqslant n} \int_{t_{\nu-1}}^{t_\nu} \|u(s)\|_{A_0} \frac{ds}{s} = b_0(t_n) = (1+\varepsilon)^{2+n} C\beta.$$

Our obvious last step is to define the function $v(t)$ to assume constant values on each of the intervals $I_\nu=((1+\varepsilon)^{\nu-1} C, (1+\varepsilon)^\nu C]$ so that $\int_{I_\nu} v(t)(dt/t)=u_{\nu+1}$. It is easy to check that v has all the required properties. ∎

(4.5) *Remark.* The preceding proof implicitly contains a "discrete" version of Theorem (4.2).

(4.6) *Remark.* In [JRW] a functional F was introduced which, in a certain sense, is dual to the E-functional and plays a similar role vis-à-vis the E-functional as the J-functional does vis-à-vis the K-functional. The F-functional of an element in $\Delta(\bar{A})$ (with respect to a Banach couple $\bar{A})=(A_0, A_1)$ is defined by

$$F(t, a; \bar{A}) = \begin{cases} \|a\|_{A_1} & \text{if } t \geqslant \|a\| \\ \infty & \text{otherwise.} \end{cases}$$

It is easy to state a version of Theorem (4.2) in terms of F: Under the same hypotheses as in Theorem (4.2). Then for each positive ε there exists a $\Delta(\bar{A})$ valued function $v = v_\varepsilon$ on $(0, \infty)$ which is strongly measurable on each subinterval (ξ, η), $\xi, \eta > 0$, such that:

(i) $a = \int_0^\infty v(t)(dt/t)$ (convergence in $\Sigma(\bar{A})$ and also in $\Sigma_E(\bar{A})$),

(ii) $\int_{t\beta(1+\varepsilon)}^\infty F(s, v(s)(ds/s) \leqslant \beta(1+\varepsilon)\, E(t, a; \bar{A})$ for all $t > 0$, where β is the same constant as appears in (2.13).

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